IMC 2022

First Day, August 3, 2022

Please do not share the problems with other people until the problems are posted on the IMC website. Thank you.

Problem 1. Let $f : [0,1] \to (0,\infty)$ be an integrable function such that $f(x) \cdot f(1-x) = 1$ for all $x \in [0,1]$. Prove that

$$\int_0^1 f(x)\ \mathrm{d}x \geq 1.$$

(10 points)

Problem 2. Let n be a positive integer. Find all $n \times n$ real matrices A with only real eigenvalues satisfying

$$A + A^k = A^T$$

for some integer $k \geq n$.

(A^T denotes the transpose of A.)

(10 points)

Problem 3. Let p be a prime number. A flea is staying at point 0 of the real line. At each minute, the flea has three possibilities: to stay at its position, or to move by 1 to the left or to the right. After $p-1$ minutes, it wants to be at 0 again. Denote by $f(p)$ the number of its strategies to do this (for example, $f(3) = 3$: it may either stay at 0 for the entire time, or go to the left and then to the right, or go to the right and then to the left). Find $f(p)$ modulo p.

(10 points)

Problem 4. Let $n > 3$ be an integer. Let Ω be the set of all triples of distinct elements of $\{1, 2, \ldots, n\}$. Let m denote the minimal number of colours which suffice to colour Ω so that whenever $1 \leq a < b < c < d \leq n$, the triples $\{a, b, c\}$ and $\{b, c, d\}$ have different colours. Prove that

$$\frac{1}{100} \log \log n \leqslant m \leqslant 100 \log \log n.$$

(10 points)

IMC 2022

Second Day, August 4, 2022

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Problem 5. We colour all the sides and diagonals of a regular polygon P with 43 vertices either red or blue in such a way that every vertex is an endpoint of 20 red segments and 22 blue segments. A triangle formed by vertices of P is called monochromatic if all of its sides have the same colour. Suppose that there are 2022 blue monochromatic triangles. How many red monochromatic triangles are there?

(10 points)

Problem 6. Let $p > 2$ be a prime number. Prove that there is a permutation $(x_1, x_2, ..., x_{p-1})$ of the numbers $(1, 2, ..., p-1)$ such that

$$x_1x_2 + x_2x_3 + ... + x_{p-2}x_{p-1} \equiv 2 \pmod{p}.$$

(10 points)

Problem 7. Let $A_1, A_2, \ldots, A_k$ be $n \times n$ idempotent complex matrices such that

$$A_iA_j = -A_jA_i \quad \text{for all } i \neq j.$$

Prove that at least one of the given matrices has rank $\leq \frac{n}{k}$.

(A matrix A is called idempotent if $A^2 = A$.)

(10 points)

Problem 8. Let $n, k \geq 3$ be integers, and let S be a circle. Let n blue points and k red points be chosen uniformly and independently at random on the circle S. Denote by F the intersection of the convex hull of the red points and the convex hull of the blue points. Let m be the number of vertices of the convex polygon F (in particular, $m = 0$ when F is empty). Find the expected value of m.

(10 points)